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On higher-order wave theory for submerged two-dimensional bodies

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The importance of non-linear free-surface effects on potential flow past two-dimensional submerged bodies is investigated by the use of higher-order perturbation theory. A consistent second-order solution for general body shapes is derived. A comparison between experimental data and theory is presented for the free-surface waves and for the wave resistance of a foil-shaped body. The agreement is good in general for the second-order theory, while the linear theory is shown to be inadequate for predicting the wave drag at the relatively small submergence treated here. It is also shown, by including the third-order free-surface effects, how the solution to the general wave theory breaks down at low speeds.

1. Introduction

The problem of a body moving in or beneath a free surface under gravity is approximated usually by linear potential flow theory assuming small surface disturbances. This is due to the intractable nature of the non-linear free-surface conditions. For most realistic problems, however, this simplified theory is far from adequate. For example, the Michell (1898) linear ship-wave theory predicts wave resistance which differs from experimental data by as much as a factor of three. The author believes that a main part of the discrepancies between analytical and experimental results could be due to the neglect of the non-linear effects at the free surface, and that the viscous effect is probably not as important as often stated. This non-linearity is investigated here, and in particular its effect on the wave resistance.

Only two investigators have applied a consistent second-order wave theory to the problem of free-surface effects on the flow past bodies: Bessho (1957) and Tuck (1965). Both restricted themselves to the simplified two-dimensional case of a submerged circular cylinder. Bessho derived correctly the complex potential, but neglected the most important higher-order term in deriving the force. His final result, and many of his conclusions, are therefore incorrect. Tuck, on the other hand, correctly obtained the wave resistance and the lift for the circular cylinder; he also correctly stated the very opposite conclusion of Bessho, namely that for a circular cylinder 'it is more important to correct for non-linearity at the free surface than for the fact that the boundary condition is not satisfied exactly by the first approximation on the body surface'.

Tuck's paper appears to be the only work to date on flow past a body in which the effect of non-linearity at the free surface has been treated correctly to the second order.†

Salvesen (1966) and Giesing & Smith (1967) investigated second-order effects on the wave resistance of two-dimensional foil-shaped bodies. They took second-order effects into account, but not in a consistent manner. Salvesen satisfied the free-surface condition correct to the second order, while the condition on the surface of the cylinder was satisfied only to the first-order approximation. Giesing & Smith, on the other hand, satisfied the body condition to second order but included only linear free-surface terms.

The main objective of the present work has been to formulate a consistent second-order theory for any general two-dimensional body, and to investigate the accuracy of this consistent theory. A comparison between experimental data and theory is presented for the free-surface elevation and the wave resistance of a foil-shaped body. It is shown that the agreement is in general good for the consistent second-order theory, but that the linear theory is inadequate for predicting the wave drag at the relatively small submergence treated here.

The results also show that at lower speeds the main higher-order contribution to the wave resistance comes from the free-surface effect, while at higher speeds the most important contribution comes from satisfying the cylinder boundary condition correct to second order. This is an interesting result, considering that, in the case of a circular cylinder, Tuck (1965) showed that the free-surface contribution was the most important for the entire speed range. The main reason for this difference is due to the circulation which is introduced for the wing-shaped body treated here in order to satisfy the Kutta condition.

It is often overlooked that the linear or higher-order wave theory generally used for free-surface flow past bodies becomes singular as the Froude number approaches zero, and that the solution breaks down for small Froude numbers. This singular behaviour of the solution at low speeds is investigated here by including not only the first- and second-order effects, but also some third-order effects.

2. Formulation of the problem

An infinitely long cylinder is supposed to move with a constant velocity U in a direction perpendicular to its axis and at a fixed distance below the undisturbed free surface. The problem is to determine the surface waves and the wave resistance.

The flow will be treated as steady in a co-ordinate system moving with the cylinder. A two-dimensional co-ordinate system will be used with the y-axis pointing upwards, and the x-axis located a distance b below the undisturbed free surface. The direction of increasing x coincides with the direction of motion of the cylinder. It will be assumed that the fluid is inviscid, incompressible and without surface tension, and that the flow is irrotational.

† Kim (1968) recently solved the second-order three-dimensional problem of a sphere below the free surface.

Formulating this two-dimensional, steady-state problem in terms of the stream function $\psi(x,y)$, we have that ψ must be a solution of the Laplace equation

$$\nabla^2\psi = \psi_{xx}+\psi_{yy} = 0, \tag{2.1}$$

and satisfy the kinematic and the dynamic free-surface conditions

$$\psi = -bU \tag{2.2}$$

and

$$\tfrac{1}{2}(\psi_y^2+\psi_x^2)+gy = \tfrac{1}{2}U^2+gb \tag{2.3}$$

on the unknown free surface, $y = b+\eta(x)$. Here g is the gravitational acceleration. On the cylinder surface we have

$$\psi = \text{constant}, \tag{2.4}$$

and the Kutta condition specifying that the trailing edge is a stagnation point. Assuming infinite depth, we have that

$$\lim_{y\to-\infty} (\operatorname{grad}\psi) = -U\mathbf{j}, \tag{2.5}$$

where $\mathbf{j}$ is the unit vector in the y-direction. In addition to these conditions, we must also specify the absence of waves far upstream.

The boundary conditions at the free surface are clearly non-linear. Assuming, therefore, that the submergence of the body is large, and that the free-surface disturbances are all small, the problem can be reformulated in terms of a perturbation scheme. In particular, we assume that the stream function ψ can be expanded in an asymptotic series in terms of the small parameter

$$\epsilon = t/b, \tag{2.6}$$

where t is the vertical dimension of the body and b is the submergence. Both the vertical and the horizontal dimension of the body are assumed to be of the same order. The near-field expansion valid close to the body can then be written as

$$\psi(x,y) = [-Uy+\psi_{B0}]+[\psi_{F1}+\psi_{B1}]+\ldots+[\psi_{Fn}+\psi_{Bn}]+\ldots, \tag{2.7}$$

where the terms in the first, second, ..., nth brackets are assumed to be of $O(1)$, $O(\epsilon)$, ..., $O(\epsilon^n)$ respectively, and where $\psi_{B0}, \psi_{B1}, \ldots, \psi_{Bn}$ are chosen so that the terms in the brackets satisfy the body conditions exactly, and $\psi_{F1}, \psi_{F2}, \ldots, \psi_{Fn}$ are the first-, second-, ..., nth-order free-surface contributions.

On the other hand, the expansion valid in the far field, and near the free surface in particular, will be assumed to be of the form

$$\psi(x,y) = -Uy+\psi^{(1)}+\psi^{(2)}+\ldots+\psi^{(n)}+\ldots, \tag{2.8}$$

with

$$\psi^{(1)} = \psi_{B0}+\psi_{F1} = O(\epsilon),$$
$$\psi^{(2)} = \psi_{B1}+\psi_{F2} = O(\epsilon^2), \text{ etc.}$$

This means that we are in fact perturbing about the uniform flow, $\psi = -Uy$, in the far field, while in the near field we are perturbing about the flow past the body in an unbounded fluid, $\psi = [-Uy+\psi_{B0}]$. In the same way we assume that the free-surface elevation $\eta(x)$ has the expansion

$$\eta(x) = \eta^{(1)}+\eta^{(2)}+\ldots+\eta^{(n)}+\ldots, \tag{2.9}$$

and that the uniform-stream velocity U is still an unknown of the problem, with the expansion

$$U = u^{(0)} + u^{(1)} + u^{(2)} + \dots + u^{(n)} + \dots, \tag{2.10}$$

where $\eta^{(n)}$ and $u^{(n)}$ are both of $O(\epsilon^n)$. The uniform-stream velocity has been chosen here as an unknown, in the manner of Wehausen & Laitone (1960, p. 655). This simplifies the derivation; but it is probably physically more attractive to let the wavelength or the wave-number be unknown, and expand these quantities.

Substituting the expansion (2.7) or (2.8) in the Laplace equation (2.1), it follows that each of the contributions to the stream function, ψ_{Bn} and ψ_{Fn}, must be a harmonic function. A substitution of the near-field expansion (2.7) in the body condition (2.4) gives that ψ_{B0} must satisfy the condition

$$[-u^{(0)}y + \psi_{B0}] = 0 \tag{2.11}$$

on the cylinder surface plus the Kutta condition. Then substitution of the far-field expansion (2.8) and the free-surface and uniform velocity expansions (2.9) and (2.10) in the free-surface conditions (2.2) and (2.3) gives the linearized free-surface condition

$$\left[\frac{\partial}{\partial y} - \nu\right][\psi_{B0} + \psi_{F1}] = 0 \quad \text{on} \quad y = b. \tag{2.12}$$

ψ_{F1} must satisfy in addition to this condition the infinity condition. Here $\nu = g/u^{(0)2}$ is the wave-number. Further substitution shows that the next-order body singularities, given by ψ_{B1}, are obtained by satisfying

$$[\psi_{F1} + \psi_{B1}] = u^{(1)}y, \tag{2.13}$$

at the cylinder wall, together with the Kutta condition, and that the second-order free-surface term ψ_{F2} must satisfy the radiation conditions and the inhomogeneous free-surface condition

$$\left[\frac{\partial}{\partial y} - \nu\right][\psi_{B1} + \psi_{F2}] = f^{(2)}(x) \quad \text{on} \quad y = b, \tag{2.14}$$

where the right-hand side is a function of the first-order solution,

$$f^{(2)}(x) = \frac{1}{2u^{(0)}}(\psi_y^{(1)2} + \psi_x^{(1)2}) + \eta^{(1)}[\nu\psi_y^{(1)} - \psi_{yy}^{(1)}] - 2\nu u^{(1)}\eta^{(1)}. \tag{2.15}$$

Similarly one can systematically obtain the conditions for any higher-order terms and in general the nth-order free-surface condition can be written as

$$\left[\frac{\partial}{\partial y} - \nu\right][\psi_{Bn-1} + \psi_{Fn}] = f^{(n)}(x) \quad \text{on} \quad y = b, \tag{2.16}$$

where of special interest is the third-order function

$$\begin{aligned} f^{(3)}(x) = {} & \nu[\eta^{(2)}\psi_y^{(1)} + \tfrac{1}{2}\eta^{(1)2}\psi_{yy}^{(1)} + \eta^{(1)}\psi_y^{(2)}] - \eta^{(2)}\psi_{yy}^{(1)} - \tfrac{1}{2}\eta^{(1)2}\psi_{yyy}^{(1)} \\ & - \eta^{(1)}\psi_{yy}^{(2)} + \frac{1}{u^{(0)}}[\eta^{(1)}\psi_y^{(1)}\psi_{yy}^{(1)} + \eta^{(1)}\psi_x^{(1)}\psi_{xy}^{(1)} + \psi_y^{(1)}\psi_y^{(2)} + \psi_x^{(1)}\psi_x^{(2)}] \\ & - 2u^{(2)}\nu\eta^{(1)} + u^{(1)}[\eta^{(1)}\psi_{yy}^{(1)} - \psi_y^{(2)} - \nu\eta^{(2)}]. \end{aligned} \tag{2.17}$$

It also follows from these substitutions that the first-, second- and third-order free-surface elevations are given by

$$\eta^{(1)} = \frac{1}{u^{(0)}} \psi^{(1)}(x, b), \tag{2.18}$$

$$\eta^{(2)} = \frac{1}{u^{(0)}} [\psi^{(2)} + \eta^{(1)} \psi_y^{(1)} - u^{(1)} \eta^{(1)}]_{y=b}, \tag{2.19}$$

$$\eta^{(3)} = \frac{1}{u^{(0)}} [\psi^{(3)} + \eta^{(2)} \psi_y^{(1)} + \eta^{(1)} \psi_y^{(2)} + \tfrac{1}{2} \eta^{(1)2} \psi_{yy}^{(1)} - u^{(2)} \eta^{(1)} - u^{(1)} \eta^{(2)}]_{y=b}. \tag{2.20}$$

It should be pointed out that in deriving the first- and higher-order free-surface conditions it was assumed that the non-dimensional wave-number, $b\nu = gb/u^{(0)2}$, is of order 1 (i.e. the Froude number, $Fr = u^{(0)}/\surd(gb)$, is also of order 1). Strictly speaking, therefore, this scheme is applicable only when the wavelength, $\lambda = 2\pi/\nu$, is of the same order as the submergence of the body.

3. Consistent second-order theory

(i) *Derivation of the stream function*

The singularity representation of any general body without circulation in an infinite flow may be written as a line integral over the cylinder contour

$$\psi_{B0}(x, y) = \frac{1}{2\pi} \operatorname{Im} \int_L m(\zeta) \ln (z - \zeta)\, d\zeta, \tag{3.1}$$

where the real source strength $m(\zeta)$ is determined by the cylinder condition (2.11). Here the complex variable $z = x + iy$ has been introduced, and Im stands for the imaginary part.

For numerical computations it is most convenient to approximate the line L by a large number of straight-line segments with constant source strength m_j over the jth segment,

$$\psi_{B0} = \frac{1}{2\pi} \operatorname{Im} \sum_{j=1}^{N} m_j \ln (z - z_j). \tag{3.2}$$

Then, satisfying condition (2.11) at the midpoint (z_j) of each segment results in N equations with N unknown source strengths, which can be solved using standard matrix techniques. If the Kutta condition applies, a concentrated vortex of strength τ at some point z_0 inside the body is introduced to produce the desired circulation. This circulation causes a flow normal to the cylinder wall, which must be cancelled by the appropriate adjustment to the source strengths, m_j. If we let $m_0 = i\tau$, we may for the circulation case write the stream function as

$$\psi_{B0} = \frac{1}{2\pi} \operatorname{Im} \sum_{j=0}^{N} m_j \ln (z - z_j). \tag{3.3}$$

From Wehausen & Laitone (1960, p. 489), the first-order free-surface term ψ_{F1}, satisfying the linearized surface condition (2.12) and the conditions at infinity, is then

$$\psi_{F1} = \frac{1}{2\pi} \operatorname{Im} \sum_{j=0}^{N} m_j \left[\ln (z - z_j^*) - 2e^{-i\nu z} \int_{\infty}^{z} \frac{e^{i\nu u}}{u - z_j^*} du \right], \tag{3.4}$$

where $z_j^* = \bar{z}_j + i2b$ (with $\bar{z}$ the conjugate of z). Define the complex function

$$I(\zeta) = e^{-\zeta} E_1(-\zeta) = e^{-\zeta} \int_{-\zeta}^{\infty} \frac{e^{-u}}{u} du, \tag{3.5}$$

where the notation of the exponential integral E_1 is that of Abramowitz & Stegun (1964); then (3.4) can be written as

$$\psi_{F1} = \frac{1}{2\pi} \operatorname{Im} \sum_{j=0}^{N} m_j [\ln(z - z_j^*) + 2I\{i\nu(z - z_j^*)\}]. \tag{3.6}$$

Introducing known expansions for E_1, it can easily be shown that the complex function (3.5) has the series expansion

$$I(\zeta) = e^{-\zeta} \left(-\gamma - \ln \zeta + i\pi - \sum_{n=1}^{\infty} \frac{\zeta^n}{n.n!} \right) \tag{3.7}$$

and the asymptotic expansion

$$I(\zeta) \sim \left\{ \frac{1}{\zeta} + \frac{1}{\zeta^2} + \frac{2!}{\zeta^3} + \frac{3!}{\zeta^4} + \dots \right\} + (1 \pm 1)\, i\pi e^{-\zeta}, \tag{3.8}$$

with plus sign for $\operatorname{Im} \zeta \to -\infty$ and minus sign for $\operatorname{Im} \zeta \to +\infty$.

The next-order singularity ψ_{B1} can be represented in the same way as ψ_{B0}, namely

$$\psi_{B1} = \frac{1}{2\pi} \operatorname{Im} \sum_{k=0}^{M} \sigma_k \ln(z - z_k), \tag{3.9}$$

where it follows from the wall condition (2.13) that the M unknown source strengths σ_k are determined by satisfying

$$\operatorname{Im} \sum_{k=1}^{M} \sigma_k \ln(z - z_k) + \operatorname{Im} \sum_{j=0}^{N} m_j [\ln(z - z_j^*) + 2I\{i\nu(z - z_j^*)\}] = 2\pi u^{(1)} y \tag{3.10}$$

at the M number of midpoints z_k. Note that (3.10) requires that the expansion (3.7) be evaluated $N \times M$ times, which can be quite time-consuming. The Kutta condition is satisfied as in the previous case by a vortex of strength σ_0 with the necessary adjustments to the source strengths, σ_k.

The next step is to obtain the second-order free-surface term, ψ_{F2}, which must satisfy the inhomogeneous surface condition (2.14). If we let

$$\psi_{F2} = {}^1\psi_{F2} + {}^2\psi_{F2}, \tag{3.11}$$

we may rewrite the free-surface condition (2.14) as

$$\left[\frac{\partial}{\partial y} - \nu \right] [\psi_{B1} + {}^1\psi_{F2}] = 0 \quad \text{at} \quad y = b, \tag{3.12}$$

and

$$\left[\frac{\partial}{\partial y} - \nu \right] [{}^2\psi_{F2}] = f^{(2)}(x) \quad \text{at} \quad y = b. \tag{3.13}$$

It now follows from (3.12) that

$${}^1\psi_{F2} = \frac{1}{2\pi} \operatorname{Im} \sum_{k=0}^{M} \sigma_k [\ln(z - z_k^*) + 2I\{i\nu(z - z_k^*)\}]. \tag{3.14}$$

Then, by recognizing that condition (3.13) is the same as the free-surface condition for the linear problem of a fixed pressure distribution $f^{(2)}(x)$ on the free surface of a uniform stream, we can apply the solution to the pressure problem given by Wehausen & Laitone (1960, p. 601). Whence we get

$$^{2}\psi_{F2} = \frac{1}{\pi u^{(0)}}\int_{-\infty}^{\infty} ds\, f^{(2)}(s)\,\mathrm{Re}\, I\{i\nu(z-s-ib)\}. \tag{3.15}$$

The solution (3.15) is bounded only if the function $f^{(2)}(x)$, given by (2.15), is non-oscillatory for large negative values of x. Applying the asymptotic expansion (3.8) to (2.15), we have

$$\lim_{x\to-\infty} f^{(2)}(x) = \text{const.} - 2\nu u^{(1)}\eta^{(1)}(x),$$

where $\eta^{(1)}(x)$ is a regular outgoing sinusoidal wave. Hence, we must set

$$u^{(1)} = 0. \tag{3.16}$$

Using the equation
$$\frac{d}{dz} I\{i\nu z\} = -\frac{1}{z} - i\nu I\{i\nu z\} \tag{3.17}$$

the function $f^{(2)}(x)$ can now be written in terms of the first-order solution (3.3) and (3.6) as

$$f^{(2)}(x) = \frac{-1}{2\pi u^{(0)}}\left|\sum_{j=0}^{N} m_j\left[\frac{\mathrm{Im}\,\xi}{|\xi|^2} + \nu I(i\nu\xi)\right]\right|^2 - \frac{1}{\pi^2}\mathrm{Im}\sum_{j=0}^{N} m_j I(i\nu\xi)\sum_{j=1}^{N} m_j\,\mathrm{Re}\,\xi\left(\frac{2\,\mathrm{Im}\,\xi}{|\xi|^4} - \frac{1}{|\xi|^2}\right), \tag{3.18}$$

where $\xi = (x-x_j)-i(b-y_j)$. This completes the second-order solution valid at the free surface, and in the far field

$$\psi(x,y) = -Uy + [\psi_{B0}+\psi_{F1}] + [\psi_{B1} + {}^{1}\psi_{F2} + {}^{2}\psi_{F2}] + O(\epsilon^3), \tag{3.19}$$

where the individual terms are given by equations (3.3), (3.6), (3.9), (3.10), (3.11), and (3.15).

The solution (3.19) is not correct to the second order in ϵ in the near field, as seen from equation (2.7). If the near-field solution is required to this order, the additional term ψ_{B2} must be determined. In this work, however, we are interested in only the wave elevation and the wave resistance, both of which can be obtained from the far-field solution (3.19).

(ii) *Wave elevation*

The wave elevation, correct to the second order in ϵ, $\eta = \eta^{(1)} + \eta^{(2)} + \ldots$, where $\eta^{(1)}$ and $\eta^{(2)}$ are given by (2.18) and (2.19), can now be written in terms of the first- and second-order solution as

$$\eta^{(1)} = \frac{1}{\pi u^{(0)}}\mathrm{Im}\sum_{j=0}^{N} m_j I\{i\nu\xi\}, \tag{3.20}$$

$$\eta^{(2)} = \frac{1}{\pi u^{(0)}}\mathrm{Im}\sum_{k=0}^{M}\sigma_k I\{i\nu\xi\} + \frac{1}{\pi u^{(0)2}}\int_{-\infty}^{\infty} ds\, f^{(2)}(s)\,\mathrm{Re}\, I\{i\nu(x-s)\} + \nu\eta^{(1)2}. \tag{3.21}$$

Here $\xi = (x-x_j)-i(b-y_j)$ and the functions I and $f(s)$ are given by (3.5) and (3.18) respectively. Far downstream we have by the asymptotic expansion (3.8) that

$$\eta^{(1)} \sim \alpha^{(1)} \cos(\nu x - \beta), \tag{3.22}$$

where the first-order wave amplitude and phase shift are given by

$$\alpha^{(1)} = |Q^{(1)}| \quad \text{and} \quad \beta = \arg Q^{(1)}, \tag{3.23}$$

with the complex number $Q^{(1)}$ given by

$$Q^{(1)} = \frac{2}{u^{(0)}} \sum_{j=0}^{N} m_j \, e^{\nu(z_j - ib)}. \tag{3.24}$$

Far downstream the second-order part becomes

$$\eta^{(2)} \sim \alpha^{(2)} \cos(\nu x - \delta) + \tfrac{1}{2}\nu\alpha^{(1)2} \cos 2(\nu x - \beta), \tag{3.25}$$

with

$$\alpha^{(2)} = |Q^{(2)}| \quad \text{and} \quad \delta = \arg Q^{(2)}, \tag{3.26}$$

where

$$Q^{(2)} = \frac{2}{u^{(0)}} \sum_{k=0}^{M} \sigma_k \, e^{\nu(z_k - ib)} + \frac{2i}{u^{(0)2}} \int_x^\infty ds \, f^{(2)}(s) \, e^{i\nu s}. \tag{3.27}$$

In obtaining (3.25) the fact that the function $f^{(2)}(x)$ tends to the constant value $(g\nu/2u^{(0)})\,\alpha^{(1)2}$ for large negative values of x was used. This constant is exactly the right magnitude to cancel the constant part of the term $\nu\eta^{(1)2}$ in (3.21).

(iii) *Wave resistance*

The 'exact'† formula for the wave resistance as derived by energy considerations in John (1949) or by the momentum theorem in Salvesen (1966) is

$$R = \frac{-\rho}{2} \int_{-\infty}^{b+\eta(x_0)} [\psi_y^2(x_0, y) + \psi_x^2(x_0, y)] \, dy + \frac{g\rho}{2} \eta^2(x_0), \tag{3.28}$$

where $\psi(x, y)$ is the 'exact' stream function, $\eta(x)$ is the 'exact' wave elevation, x_0 denotes any vertical plane behind the body, and ρ is the mass density of the fluid.

Observing that the wave elevation far downstream correct to $O(\epsilon^2)$ is given by (3.22) and (3.25) as

$$\eta = \alpha^{(1)} \cos(\nu x - \beta) + \tfrac{1}{2}\nu\alpha^{(1)2} \cos 2(\nu x - \beta) + \alpha^{(2)} \cos(\nu x - \delta) \tag{3.29}$$

and that the stream function far downstream, correct to the same order, is

$$\psi = u^{(0)} e^{\nu(b-y)} [\alpha^{(1)} \cos(\nu x - \beta) + \alpha^{(2)} \cos(\nu x - \delta)], \tag{3.30}$$

and evaluating the expression (3.28) at some far downstream section, it can be shown that the wave resistance, correct to the third order in ϵ, is

$$R = \tfrac{1}{4}\rho g [\alpha^{(1)2} + 2\alpha^{(1)}\alpha^{(2)} \cos(\delta - \beta)], \tag{3.31}$$

where the first- and second-order wave amplitudes, $\alpha^{(1)}$ and $\alpha^{(2)}$, and the phase shifts, δ and β, are given by (3.23) and (3.26).

† 'Exact' in quotation marks refers to exact within potential-flow theory.

Expressing the resistance in terms of the trough-to-crest wave height, which by (3.29) is correct to $O(\epsilon^2)$,

$$H = \eta\left(\frac{\beta}{\nu}\right) - \eta\left(\frac{\beta}{\nu}+\pi\right) = 2\alpha^{(1)} + 2\alpha^{(2)}\cos(\delta-\beta), \tag{3.32}$$

we have

$$R = \tfrac{1}{4}\rho g(\tfrac{1}{2}H)^2 + O(\epsilon^4). \tag{3.33}$$

Hence, using this result with experimental work, where H is the actual measured downstream trough-to-crest wave height, equation (3.33) gives the wave resistance correct to $O(\epsilon^3)$ even when the waves are not exactly sinusoidal.

(iv) *Numerical and experimental results*

In order to investigate the importance of the second-order terms and the improved accuracy obtained by including them, the wave elevation and wave resistance for a given body were both numerically computed on an IBM 7090 computer and experimentally obtained.

It was recognized that a considerable saving in computation time could be achieved by selecting a body whose singularity representation in an unbounded uniform flow is of a simple known form. A symmetrical body represented in a uniform flow by eleven concentrated sources equally spaced along the line of symmetry was chosen:

$$\psi_{B0} = \frac{1}{2\pi}\sum_{j=0}^{10} m_j \tan^{-1}\frac{y}{x-j/10}, \tag{3.34}$$

with the known source strengths

$$\left.\begin{aligned} m_j &= -0{\cdot}08\pi U(5+8j/7)/24 \quad &&\text{when} \quad 0 \leqslant j \leqslant 7, \\ &= +0{\cdot}08\pi U &&\text{when} \quad 8 \leqslant j \leqslant 10. \end{aligned}\right\} \tag{3.35}$$

The body-surface co-ordinates are then given by

$$y = \pm\sum_{j=0}^{10}\frac{m_j}{2\pi U}\tan^{-1}\frac{y}{x-j/10} \tag{3.36}$$

and the cross-section of the body is seen in figure 1. The first-order solution is then

$$\psi^{(1)} = \psi_{B0} + \psi_{F1}, \tag{3.37}$$

with ψ_{F1} given by (3.6) in terms of the known singularities m_j.

An additional distribution of 138 source segments on the cylinder wall and a concentrated vortex were used to represent the body correct to order ϵ^2:

$$\psi_{B1} = \frac{1}{2\pi}\operatorname{Im}\sum_{k=0}^{138}\sigma_k \ln(z-z_k), \tag{3.38}$$

where the strengths were obtained by satisfying numerically the condition (3.10) and the Kutta condition. One should note that the strengths σ_k are speed-dependent and must be recomputed for each speed of interest, while the m_j in (3.34) are independent of the speed except for the factor U in (3.35).

With the known first-order and the computed second-order singularity representations, the wave elevation correct to second order is obtained by (3.20) and (3.21) and the wave resistance by (3.31). It is especially true when computing the integral term in the second-order wave (3.21) that essential time saving is obtained by the simple form of the first-order singularity representation (3.34).

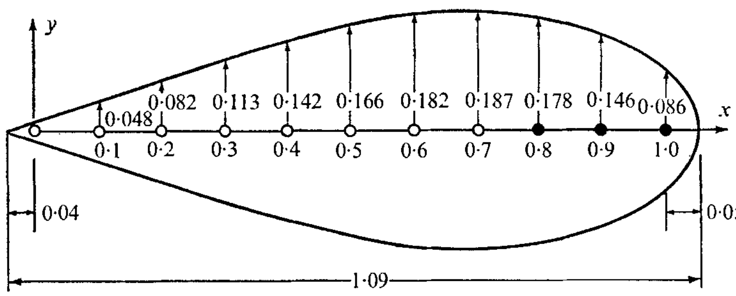

FIGURE 1. Cross-section of body.

The experiments were conducted in the main tank ($360 \times 20 \times 9$ ft.) at the Ship Hydrodynamics Laboratory of The University of Michigan. In order to represent the two-dimensional flow adequately, an 11 ft. strut with end-plates and cross-section as shown in figure 1 was used (chord length = 1·09 ft. and thickness = 0·374 ft.). The wave elevation was measured by a 0·0016 in. capacitance wire and the horizontal drag force was measured on the middle 2 ft. section of the model by water-proofed strain-gauges mounted on two cantilever beams. A more detailed description of the experiments can be found in Salvesen (1966), where many experimental results are also presented.

Since the non-linear free-surface effects are most important at the smaller submergences, comparisons will be made here between the experiments and the consistent second-order theory only at the smallest submergence for which there was no wave-breaking (b = 1·25 ft.). Wave profiles are shown in figure 2 for three speeds, Froude number equal to 0·40, 0·71 and 0·87.† These cases were selected as representative samples showing the essential features for each of the following speed ranges: very low speeds, intermediate speeds and higher speeds. At the very low speeds (Fr = 0·40) it is seen that the second-order wave height is several times the height predicted by the linear theory, which seems to violate the assumption of a converging perturbation series. However, the agreement between experiment and the second-order theory is surprisingly good at this low speed despite this violation. In the intermediate speed range (Fr = 0·71) the second-order contribution is relatively small and the agreement in wave height is very good, while the measured wavelength is seen to be about 10 % smaller than that predicted by the theory. Similar discrepancies in wavelength were found for every speed tested in this intermediate speed range. At the higher speed (Fr = 0·87), which is in the speed range of maximum wave height, and hence maximum wave resistance, quite good agreement between second-order theory and experiment

† The Froude number is here defined with respect to body submergence b, so that $Fr = U/\sqrt{(gb)}$.

is seen with respect to both wave height and wavelength. However, it is discouraging to note that at this speed the linear theory over-predicts the wave height by as much as 20 %.

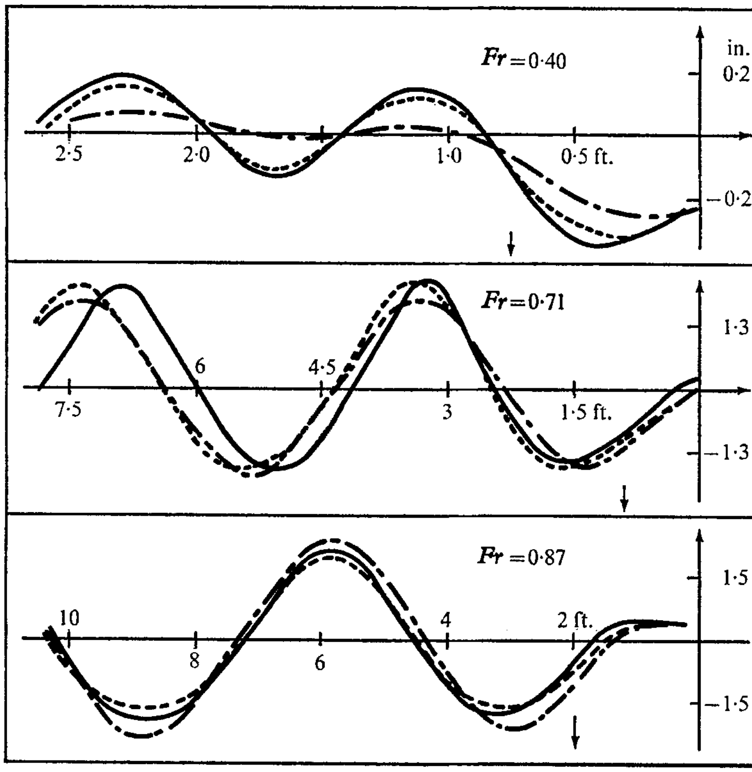

FIGURE 2. Wave-election curves for $\epsilon = t/b - 0{\cdot}30$. —·—, first-order theory; – – –, second-order theory; ———, measured wave. ↓ indicates location of trailing edge.

Figure 3 shows the wave-resistance curves from first- and second-order theory and from experiments. There are two experimental curves. One, the solid line, is obtained from wave-height measurements, using the derived equation (3.33), $R = \frac{1}{4}\rho g(\frac{1}{2}H)^2$. The other, the dotted line, is obtained from drag measurements by subtracting the horizontal drag at 4·5 ft. submergence from the total drag at the 1·25 ft. submergence (assuming only viscous drag at the 4·5 ft. submergence and no interaction between wave and viscous resistance). The wave resistance from the drag measurements is rather high at the lower speeds, which seems to indicate that the viscous drag is not the same at 4·5 and 1·25 ft. submergence, but that it increases as the body gets closer to the free surface. This is most likely due to an increase in the velocity of the fluid next to the body as a result of the free surface. Furthermore, this shows how extremely difficult it is to determine the actual wave resistance from drag-force measurement. The author strongly believes, therefore, that, when checking the validity of theoretical work, one should not use drag force data, but rather the data from wave survey.

Comparing the wave resistance from wave survey with the theory in figure 3, it is seen that the linear theory agrees rather poorly with the experiment. It grossly underestimates the wave resistance at the lower speeds and overestimates it by as much as a factor of two at the higher speeds. On the other hand, the consistent second-order theory shows fair agreement with experiment over the entire speed range.

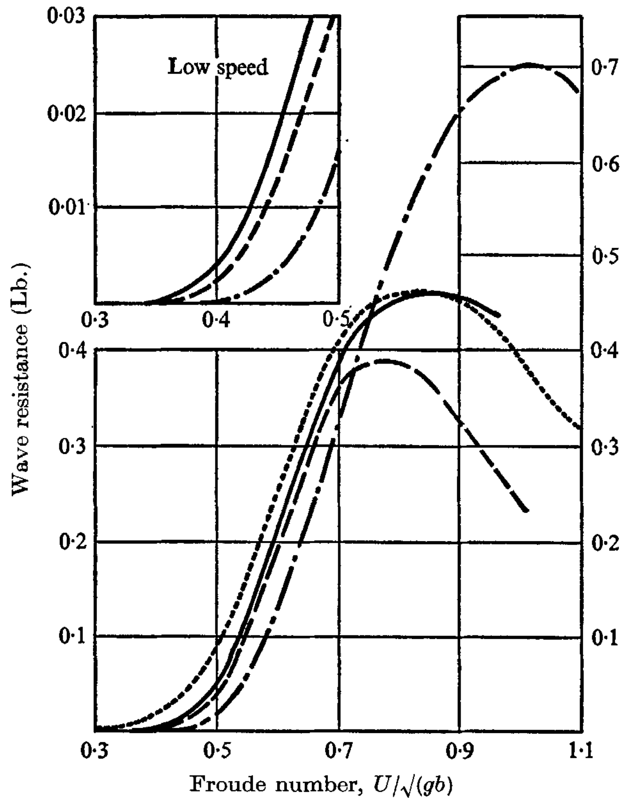

FIGURE 3. Wave-resistance curves for $\epsilon = t/b = 0{\cdot}30$. —·—, first-order theory; – – –, second-order theory; ———, from measured wave; - - - -, difference between horizontal drag at 1·25 and 4·5 ft. submergence.

This substantial difference between the linear and the second-order theory should be considered in terms of the two different second-order contributions: (a) the second-order linear body correction effect, and (b) the second-order non-linear free-surface effect. Figure 4 has been prepared in order to separate these two effects. The figure shows the wave resistance obtained by (a) linear theory, (b) inconsistent second-order theory (neglecting the body correction effects), and (c) consistent second-order theory. Comparison of these curves shows that both second-order terms are important and that neither should be neglected. It is especially interesting to note that at lower speeds ($Fr < 0{\cdot}65$) the main higher-order contribution to the wave resistance comes from the free-surface effect; however, at higher speeds ($Fr > 0{\cdot}75$) it is seen that the body-correction term gives the essential contribution. This is an interesting result, considering that in the case of the circular cylinder Tuck (1965) showed that the free-surface contribution was the most important for the entire speed range. The reason that the body-correction effect is so important for the wing-shaped body treated here is that, in addition to the singularities introduced in closing the body, a circulation term is introduced such that the Kutta condition is satisfied. The numerical results show that it is mainly this circulation term which gives rise to the large second-order effect at higher speeds ($Fr > 0{\cdot}75$).

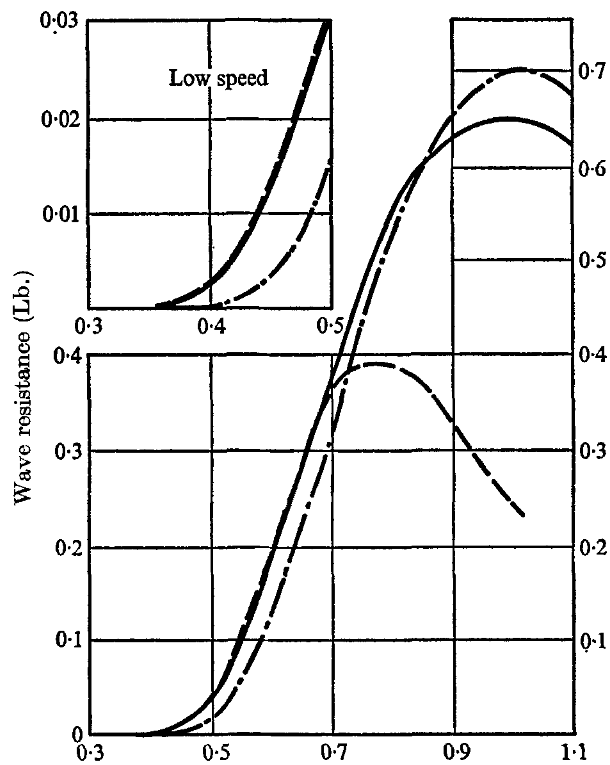

FIGURE 4. Theoretical wave-resistance curves for $\epsilon = t/b = 0{\cdot}30$. —·—, first-order theory; ———, inconsistent second-order (neglecting body-correlation effects); – – –, consistent second-order.

It should be emphasized that the free-surface disturbances are quite severe in the case presented here and that the ratio between the chord length and the submergence is not small but almost equal to one. This severe case was selected so that a better comparison could be made of the relative importance of the different second-order contributions. As the submergence increases, however, the second-order effects will become less dominating and also the agreement with the experiment will improve greatly.

In general, it can be stated that the consistent second-order theory agrees quite well with the experiments; but there are two aspects of the results which are less satisfactory. First, at the very low speeds, the second-order contribution is several times the linear one, and secondly, in the intermediate speed range, both the linear and the second-order theory seem to over-predict the wavelength. The next section is devoted mainly to the study of these two interesting points.

4. Third-order effects

In deriving the first-, second- and higher-order free-surface conditions,

$$\psi_y^{(i)} - \nu\psi^{(i)} = f^{(i)}(x) \quad \text{at} \quad y = b, \tag{4.1}$$

it was necessary to assume that the wave-number ν is $O(1)$ so that $\nu\psi^{(i)} = O(\epsilon^i)$. Clearly this expansion scheme becomes invalid as $U \to 0$ and $\nu \to \infty$. In fact, the expansion is singular for $U = 0$.

Unfortunately, very little is known about how the theory breaks down as $U \to 0$, and for a given case it is impossible to tell *a priori* in what range of values of the non-dimensional wave-number, $\nu b = gb/U^2$, the results become invalid.

This behaviour will be studied here by computing the wave profile correct to the third order in ϵ and by investigating the convergence of the first three terms for low speeds. A given singularity distribution will be used as the submerged disturbance rather than a solid body, since this greatly simplifies the computations by eliminating the body-boundary condition. This approach should result in no loss of generality as far as the study of the convergence and the validity of the series expansion at low speeds are concerned.

Assume (as in § 2) that the stream function and the uniform-stream velocity have the following expansions valid near the free surface and in the far field:

$$\psi(x,y) = -Uy + \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \ldots, \tag{4.2}$$

$$U = u^{(0)} + u^{(1)} + u^{(2)} + \ldots. \tag{4.3}$$

Let the disturbance be represented by some singularities,

$$\psi_B = \frac{1}{2\pi} \operatorname{Im} \sum_{j=0}^{N} m_j \ln(z - z_i). \tag{4.4}$$

Then we find, by (3.3) and (3.6), that the first-order stream function is

$$\psi^{(1)} = \psi_B + \psi_F, \tag{4.5}$$

where
$$\psi_F = \frac{1}{2\pi} \operatorname{Im} \sum_{j=0}^{N} m_j \{\ln(z - z_j^*) + 2I[i\nu(z - z_j^*)]\}. \tag{4.6}$$

Since the higher-order terms $\psi^{(2)}$ and $\psi^{(3)}$ do not, in this case, have to satisfy any body condition, but only the non-linear free-surface condition (4.1), we have that the second- and third-order stream functions are simply given by

$$\psi^{(i)} = \frac{1}{\pi u^{(0)}} \int_{-\infty}^{\infty} ds f^{(i)}(s) \operatorname{Re} I\{i\nu(z - s - ib)\} \quad (i = 2, 3), \tag{4.7}$$

where $f^{(2)}(x)$ and $f^{(3)}(x)$ are expressed in (2.15) and (2.17) respectively, in terms of the lower-order solutions.

Having obtained the stream function correct to the third order for the given singularity distribution (4.4), the first-, second- and third-order terms in the wave-profile expansion,

$$\eta = \eta^{(1)} + \eta^{(2)} + \eta^{(3)} + \ldots, \tag{4.8}$$

are easily obtained by substitution of the stream function into (2.18), (2.19) and (2.20).

To ensure that the solutions (4.7) are bounded, however, the functions $f^{(i)}(x)$ must be non-oscillatory for large negative values of x. For the second-order case this requires that $u^{(1)} = 0$, as stated in (3.16). For the third-order case it can be shown that

$$\lim_{x\to-\infty} f^{(3)}(x) = \frac{g}{u^{(0)}}\,\nu\alpha^{(1)}\alpha^{(2)} + \frac{g}{u^{(0)}}\,\nu^2\alpha^{(1)2}\eta^{(1)}(x) - 2\nu u^{(2)}\eta^{(1)}(x), \tag{4.9}$$

where $\eta^{(1)}(x)$ is a regular outgoing sinusoidal wave. Hence, we must set

$$u^{(2)} = \tfrac{1}{2}\nu^2\alpha^{(1)2}u^{(0)}, \tag{4.10}$$

so that

$$\lim_{x\to-\infty} f^{(3)}(x) = \frac{g}{u^{(0)}}\,\nu\alpha^{(1)}\alpha^{(2)} = \text{const.} \tag{4.11}$$

It follows from (4.10) that the uniform-stream velocity (4.3) is

$$U = u^{(0)}(1 + \tfrac{1}{2}\nu^2\alpha^{(1)2}), \tag{4.12}$$

which we recognize as the result originally obtained by Stokes (1847) for the third-order Stokes wave. This implies that the wavelength according to the third-order theory is

$$\lambda = \frac{2\pi}{\nu} = \frac{2\pi}{g}\,U^2(1 - \nu^2\alpha^{(1)2}), \tag{4.13}$$

while the wavelength given by both the linear and the second-order theory is $\lambda = (2\pi/g)\,U^2$.

Let us now turn to some of the numerical results. The same eleven concentrated sources (3.35) which represent the tested body (figure 1) in an unbounded uniform flow were used as the submerged disturbance. The first-, second- and third-order contributions to the far downstream wave heights ($H^{(1)}$, $H^{(2)}$ and $H^{(3)}$) were computed in the low-speed range, and the results are presented in figure 5 as the ratios

$$\frac{H^{(i)}}{H^{(1)} + H^{(2)} + H^{(3)}} \quad (i = 1, 2 \text{ and } 3), \tag{4.14}$$

plotted against the non-dimensional number $\epsilon\nu b = gt/u^{(0)2}$.† One notes in this figure the reversal of the relative magnitude of the first-, second-, and third-order terms as the speed decreases. This is typical behaviour for perturbation solutions approaching a singular point, in this case $U = 0$. The very same behaviour can be seen, for example, in Van Dyke (1964) for the singular perturbation problem of a 'slightly supersonic flow past a slender circular cone'.

† Note that $gt/u^{(0)2}$ is actually a wave-number rendered non-dimensional by the body thickness, making these results independent of the body submergence, and that t is here redefined as the vertical dimension of the body created by the given singularity distribution in an unbounded uniform flow.

The low-speed case, $Fr = 0{\cdot}40$ and $\epsilon = t/b = 0{\cdot}30$, for which the first- and consistent second-order profiles were presented in figure 2, is also marked in figure 5. For this case it is seen that, while the second-order wave height is much larger than the first-order, the third-order contribution is relatively 'small'. This strongly suggests the possibility that the terms in the expansion start to diminish after the second-order term and that the series is most likely convergent even at this speed. The good agreement between experiment and consistent second-order theory, as shown in figure 2, also seems to support this possibility.

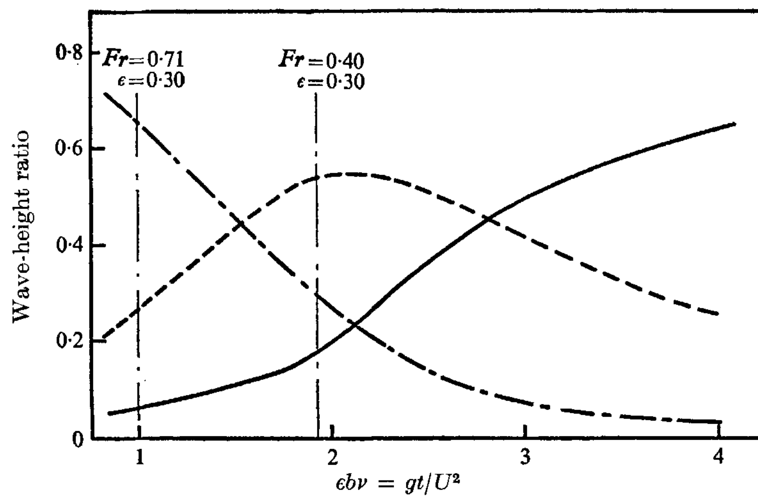

FIGURE 5. First-, second- and third-order wave heights at low speeds. —·—, $H^{(1)}/(H^{(1)}+H^{(2)}+H^{(3)})$; - - -, $H^{(2)}/(H^{(1)}+H^{(2)}+H^{(3)})$; —, $H^{(3)}/(H^{(1)}+H^{(2)}+H^{(3)})$.

It seems reasonable to assume that the trend of these results is applicable generally to any submerged two-dimensional body, and that one may conclude from figure 5 that the linear theory should not be applied at the very small speeds $gt/U^2 < 1{\cdot}0$. On the other hand, in the low-speed range $(1{\cdot}0 < gt/U^2 < 2{\cdot}0)$ the second-order theory may be used even if the linear theory by itself is not applicable.

The first-, second- and third-order wave profiles created by the selected submerged singularities (3.35) were computed for several combinations of speeds and submergences. Figures 6 and 7 show the cases $Fr = 0{\cdot}40$ and $Fr = 0{\cdot}71$, both with $\epsilon = 0{\cdot}30$, which are the same Froude numbers for which the consistent second-order results were presented in figure 2. For the low-speed case (figure 6) it is seen clearly that the third-order contribution is relatively small. For the intermediate speed case (figure 7) one notes the substantial third-order effect on the wavelength as previously stated in (4.13).

In figures 6 and 7 the experimental results for these Froude numbers have also been included, even though the physical problem is not correctly represented to the third order in ϵ. In fact, by selecting the singularities (3.35) which represent the body in an unbounded uniform stream, the body condition is satisfied only to the first order. However, the consistent second-order results presented in figure 4 show that for the low speeds ($Fr < 0{\cdot}70$), and especially for the very low speeds ($Fr < 0{\cdot}50$), the contribution from the second-order body correction is extremely small relative to the second-order free-surface contribution. Thus, assuming that the third-order body-correction term is also negligible, a comparison between these theoretical results and the experiment seems justified.

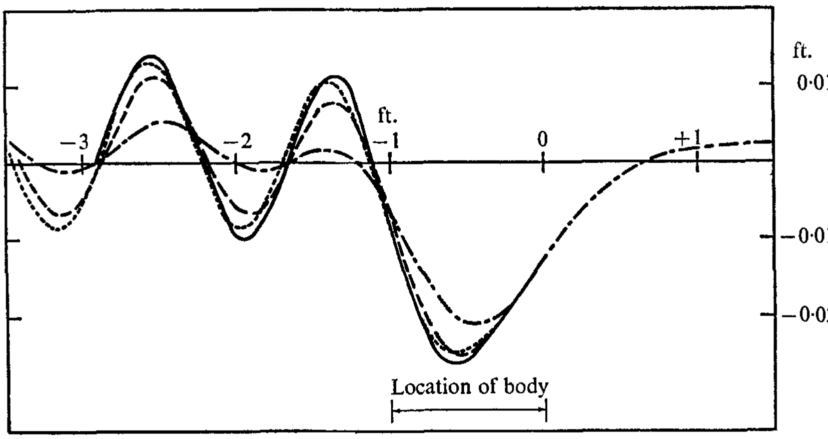

FIGURE 6. Higher-order wave profiles at low speed. —·—, first-order; — —, second-order; - - - -, third-order; ———, experiment. $Fr = 0{\cdot}40$, $\epsilon = t/b = 0{\cdot}30$.

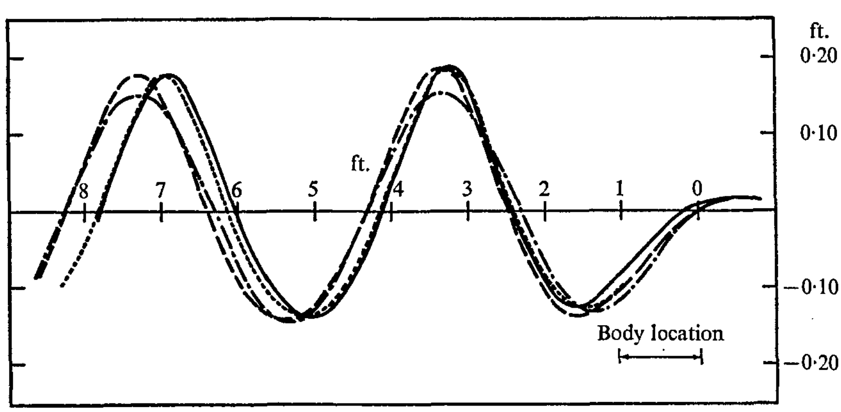

FIGURE 7. Third-order effect on wave length. —·—, first-order; – – –, second-order; - - - -, third-order; ———, experiment. $Fr = 0{\cdot}71$, $\epsilon = t/b = 0{\cdot}30$.

To apply any of these results to the three-dimensional problem of a surface ship would most likely result in incorrect statements. Because of the many similarities between the two- and three-dimensional case, however, it is tempting to make some speculations. In the linear ship-wave theory by Michell (1898) the same perturbation scheme is used with the expansion parameter $\epsilon = B/L$ (beam/length), and here the non-dimensional wave-number $L\nu = Lg/U^2$ is, by assumption, also of order 1. The solution is singular for $U = 0$; furthermore, the wave resistance obtained by the Mitchell theory agrees well with experiments at Froude numbers larger than about 0·32, while at lower speeds the linear theory is off by as much as a factor of three. (See, for example, Havelock (1951) and Inui (1957).) One notes that this limiting Froude number corresponds to a value of about 1 for the non-dimensional number $\epsilon L\nu = Bg/U^2$ in the ship case where $\epsilon \sim 0{\cdot}10$. This is about the same limiting value found for the two-dimensional problem. This seems to indicate that the inaccuracy of the linear Michell theory at $Fr < 0{\cdot}32$ may not be due to viscous effects, as so often stated, but rather to the singular behaviour of the perturbation solution at these low speeds. Taking a step further, one may speculate that, as in the two-dimensional case, there is a speed range (approximately $1{\cdot}0 < \epsilon L\nu < 2{\cdot}0$) in the case of the ship, and that the second-order theory could give reasonable results in the ship case, even if the linear theory is not applicable. Interestingly enough, the speed range corresponds approximately to $0{\cdot}22 < Fr < 0{\cdot}32$, for the ship case of $\epsilon = 0{\cdot}10$ (the practical speed range for modern commercial ships).

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